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# Gauge transformations of spinors within a Clifford algebraic structure

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**Abstract.** Algebraic spinors can be defined as minimal left ideals of Clifford algebras. We consider gauge transformations which are two-sided equivalence transformations of a complete algebra, including the spinors. These transformations of the spinors introduce new interaction terms which appear hard to interpret. We establish algebraic theorems which allow these new interaction terms to be evaluated and use these ideas to provide a new formulation of Glashow's electroweak interactions of leptons. The theorems also lead us to propose a new Clifford algebraic definition of spinors based on nilpotents, rather than idempotents.

## 1. Minimal left ideals as spinors

Clifford, or geometric, algebras have come to be recognized as providing an appropriate, and potentially rich, mathematical description of physics. The Pauli and Dirac matrices used in models of fundamental particles are matrix representations of the basis vectors of Clifford algebras associated with three-dimensional space and four-dimensional spacetime, respectively. There has been much discussion in the literature about the representation of Dirac spinors using Clifford algebras. It has become standard practice to identify spinors with minimal left ideals of the algebras [1, 2]; we shall review these ideas briefly in this section.

Let us consider the general non-degenerate Clifford algebra associated with an $n$-dimensional flat manifold. An orthonormal vector basis $\{e_\mu(x)\}$ for the algebra is defined by

$$\{e_\mu(x), e_\nu(x)\} = 2I\eta_{\mu\nu}(x) \qquad \mu, \nu = 0, 1, 2, \ldots, (n-1) \tag{1.1}$$

where $\eta_{\mu\nu}$ are the components of the metric tensor on the space with

$$\eta = \begin{pmatrix} \underbrace{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}_{p} & & \\ & \underbrace{\begin{matrix} -1 & & \\ & \ddots & \\ & & -1 \end{matrix}}_{q} & \\ & & \end{pmatrix}; \tag{1.2}$$

so

$$\eta_{\mu\mu} = 1 \qquad \mu = 0, 1, \ldots, (p-1) \tag{1.3}$$

$$\eta_{\mu\mu} = -1 \qquad \mu = p, (p+1), \ldots, (p+q-1) \tag{1.4}$$

$$\eta_{\mu\nu} = 0 \qquad \mu \neq \nu. \tag{1.5}$$

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The basis vectors $\{e_\mu; \mu = 0, 1, \ldots, p-1\}$ of the flat space are timelike and the remaining $(n-p)$ basis vectors $\{e_\mu; \mu = p, p+1, \ldots, n-1\}$ are spacelike. The set $\{e_\mu; \mu = 0, 1, \ldots, n-1\}$ forms an orthonormal vector basis of the Clifford algebra $R_{p,q}$, $p+q=n$. The complete set of basis elements of $R_{p,q}$ is comprised of $2^n$ multivector elements. We shall denote a general multivector basis element by $e_T$. Thus, $e_T$ either denotes the scalar or is a shorthand way of writing

$$e_T = e_{\mu_1} e_{\mu_2} \ldots e_{\mu_m} \equiv e_{\mu_1 \mu_2 \ldots \mu_m} \qquad \mu = 0, 1, \ldots, n-1 \quad \text{and} \quad \mu_1 < \mu_2 < \ldots < \mu_m. \tag{1.6}$$

A general element $A$ of $R_{p,q}$ is a linear combination over the reals of the basis elements:

$$A = \sum_{T=1}^{2^n} a_T e_T, \, a_T \in R. \tag{1.7}$$

Ideals of the algebra $R_{p,q}$ may be defined using idempotents which are elements $U$ of the algebra satisfying

$$U^2 = U. \tag{1.8}$$

A left ideal of the algebra $R_{p,q}$ is a subset of the algebra which is defined by the action of an idempotent $U$ on the right of each of the elements $A$ of $R_{p,q}$. Thus, a left ideal is the set

$$\{AU; \text{any } A \in R_{p,q}\}. \tag{1.9}$$

A left ideal of $R_{p,q}$ is minimal if $U$ is a primitive idempotent [1]: that is, if $U$ is the product of a maximal number $k$ of 'simple idempotents':

$$U = \frac{1}{2}(I + p_1)\frac{1}{2}(I + p_2) \ldots \frac{1}{2}(I + p_k) \equiv \frac{1}{2^k} \prod_{i=1}^{k} (I + p_i) \tag{1.10}$$

where the set of 'projectors' $\{p_i; i = 1, 2, \ldots, k\}$ is such that $p_i^2 = I$ and $[p_i, p_j] = 0$ for $i, j = 1, 2, \ldots k$. These last two conditions ensure that $U$ is an idempotent.

The value of $k$ is defined uniquely for each $R_{p,q}$ as

$$k = q - r_{q-p} \tag{1.11}$$

where $r_i$ is the *Radon–Hurwitz number*. Since the values of the Radon–Hurwitz number form a cycle of period eight:

$$r_{i+8} = r_i + 4 \tag{1.12}$$

all the values of all $r_i$ are defined by table 1. The number $k$ defined by (1.11) and (1.12) is the maximal number of commuting elements of square $I$ which can be found in any algebra $R_{p,q}$.

It has been shown [1, 3] that an element of a minimal left ideal in $R_{1,3}$ contains eight real or four complex parameters, which is sufficient to describe all the physically relevant information of a Dirac column spinor. Thus minimal left ideals in $R_{1,3}$ are normally regarded as equivalent to Dirac column spinors. This may be generalized [3] to any algebra $R_{p,q}$ in which 'algebraic spinors' are defined as elements of a minimal left ideal. In $R_{p,q}$ a basis for a minimal left ideal contains $2^{n-k}$ elements. Thus an algebraic spinor in $R_{p,q}$ contains $2^{n-k}$ real parameters, which is sufficient to describe $2^{n-k-3}$ Dirac spinors.

**Table 1.** The values of the Radon–Hurwitz number.

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $r_i$ | 0 | 1 | 2 | 2 | 3 | 3 | 3 | 3 |

We define a bar conjugate spinor from a spinor using an operation referred to as 'Hermitian conjugation' since it is equivalent to Hermitian conjugation of the matrix representation of certain algebras. The Hermitian conjugate of a basis multivector $e_T$ is defined to be

$$e_T^\dagger = e_T \qquad \text{if} \quad e_T^2 = I \tag{1.13a}$$

$$e_T^\dagger = -e_T \qquad \text{if} \quad e_T^2 = -I. \tag{1.13b}$$

In some algebras the operation of Hermitian conjugation is equivalent to an inner automorphism of the algebra [3], which is defined on any multivector basis element by

$$e_T^\dagger = \Gamma e_T^R \Gamma^{-1} \tag{1.14}$$

where $e_T^R$ denotes the reversal of $e_T$, that is

$$e_T^R = (-1)^{m(m-1)/2} e_{\mu_1\mu_2\ldots\mu_m} = (-1)^{m(m-1)/2} e_T. \tag{1.15}$$

The operator $\Gamma$ as defined in (1.14) does not exist in every $R_{p,q}$. It can be defined in algebras in which either $p$ is odd or $q$ is even, $q \neq 0$. In the first case, the operator $\Gamma$ is the $p$-vector $e_{012\ldots(p-1)}$ which is the product of the $p$ timelike basis vectors, and in the second case it is the $q$-vector $e_{p(p+1)\ldots(p+q-1)}$ which is the product of the $q$ spacelike vectors. If *both* $p$ is even *and* $q$ is odd, then $\Gamma$ does not exist.

If we denote an algebraic spinor by $\psi$, so that $\psi$ is an element of a minimal left ideal of $R_{p,q}$, then we define its bar conjugate to be

$$\bar{\psi} = \psi^\dagger \Gamma \tag{1.16}$$

which by using (1.13) can be shown to be

$$\bar{\psi} = \Gamma \psi^R. \tag{1.17}$$

To conclude this section, we consider as an example the Dirac algebra $R_{1,3}$ in which, from (1.11) and table 1, the value of $k$ is given by

$$k = 3 - r_2 = 1.$$

Therefore a primitive idempotent in $R_{1,3}$ will contain just one factor; for example,

$$U = \tfrac{1}{2}(I + e_0) \tag{1.18}$$

is primitive. Using this idempotent, we define a minimal left ideal with elements

$$\{A\tfrac{1}{2}(I + e_0); \text{ any } A \in R_{1,3}\}. \tag{1.19}$$

In this representation, the bar conjugate to this spinor is

$$\begin{aligned}\bar{\psi} &= \Gamma \psi^R = e_0 \tfrac{1}{2}(I + e_0) A^R \\ &= \tfrac{1}{2}(I + e_0) A^R.\end{aligned} \tag{1.20}$$

When the 16 basis elements of $R_{1,3}$ are post-multiplied by the primitive idempotent (1.18), we obtain eight distinct terms:

$$\{U, e_i U, e_{ij} U, e_{123} U; i, j = 1, 2, 3, i < j\}$$

which form a basis for the minimal left ideal. Thus $\psi$ and $\bar{\psi}$ may be written, respectively, as

$$\psi = (\alpha + \alpha_i e_i + \alpha_{ij} e_{ij} + \alpha_{123} e_{123}) U \qquad i, j = 1, 2, 3, \quad i \neq j \tag{1.21a}$$

and

$$\bar{\psi} = U(\alpha + \alpha_i e_i - \alpha_{ij} e_{ij} - \alpha_{123} e_{123}) \tag{1.21b}$$

where the coefficients $\alpha_T$ are real.

In matrix representation the 'Dirac metric' or 'invariant length' of the spinor is defined to be Trace($\bar{\psi}\psi$). In algebraic terms this is equivalent to determining the scalar part of the element ($\bar{\psi}\psi$) and is denoted by $\langle\bar{\psi}\psi\rangle_S$. The scalar part is equivalent to the trace of a matrix, and hence it satisfies a cyclic property

$$\langle ABC\rangle_S = \langle BCA\rangle_S = \langle CAB\rangle_S. \tag{1.22}$$

We shall calculate this quantity in the algebra $R_{1,3}$ using the ideal (1.19). The scalar part of ($\bar{\psi}\psi$) consists only of those terms in which $e_T^R$ multiplies $e_T$, that is, from the multiplication of 'like' terms. Consequently we deduce that

$$\begin{aligned}\langle\bar{\psi}\psi\rangle_S &= \langle U(\alpha^2 + \alpha_i^2 - \alpha_{ij}^2 - \alpha_{123}^2)U\rangle_S \\ &= \langle U^2(\alpha^2 + \alpha_i^2 - \alpha_{ij}^2 - \alpha_{123}^2)\rangle_S \\ &= \tfrac{1}{2}(\alpha^2 + \alpha_i^2 - \alpha_{ij}^2 - \alpha_{123}^2).\end{aligned}$$

It thus consists of a sum of eight squares, four each of positive and negative signs, which is exactly what we would expect from the product of a Dirac spinor and its bar conjugate.

Other bispinor densities including the Dirac metric are defined [4] from the expression

$$\langle\bar{\psi}B\psi\rangle_S \tag{1.23}$$

where $B$ is any element of the algebra $R_{1,3}$. For example, when $B = e_\mu(\mu = 0, 1, 2, 3)$ we define a component of the conserved current $j_\mu$ in Minkowski spacetime.

## 2. Gauge transformations of spinors

The standard Yang–Mills gauge transformation of a column spinor $\psi$ is given by

$$\psi \to Q(x)\psi \tag{2.1a}$$

$$\bar{\psi} \to \bar{\psi}Q^{-1}(x) \tag{2.1b}$$

where $Q(x)$ is some local unitary transformation. Invariance of the Lagrangian density under the local transformations (2.1) is achieved by introducing a covariant derivative $D_\mu$, where

$$D_\mu = \partial_\mu - \Omega_\mu. \tag{2.2}$$

The connection terms $\Omega_\mu$ in $D_\mu$ transform according to

$$\Omega_\mu \to Q(x)\Omega_\mu Q^{-1}(x) + (\partial_\mu Q)Q^{-1} \tag{2.3}$$

so that

$$D_\mu\psi \to QD_\mu\psi \tag{2.4}$$

and then the Lagrangian density

$$\bar{\psi}\mathrm{i}e^\mu(D_\mu\psi) \tag{2.5}$$

is gauge invariant. The term $\bar{\psi}\mathrm{i}e^\mu\Omega_\mu\psi$ in (2.5) represents the interaction of the spinor with the vector boson fields. In the algebra $R_{1,3}$ we are most familiar with, the Lagrangian density is written in the form $\bar{\psi}\mathrm{i}\gamma^\mu(D_\mu\psi)$ where $\gamma^\mu$ is the Dirac matrix representation of $e^\mu$.

The form (2.1) and (2.3) of the gauge transformation is appropriate for column spinors and conjugate row spinors. In previous papers, we have made the assumption that the form of the gauge transformations is the same for algebraic spinors, so that the transformed algebraic spinor remains within the ideal (1.9). Then the gauge transformations (2.1) of spinors are of the same 'one-sided' form as, say, transformations under the rotation group which correspond

to the half-odd-integral spin of fermions. However, the assumption that algebraic spinors undergo a transformation of the form (2.1) implies that the idempotent $U$ in the spinor

$$\psi = AU \tag{2.6}$$

transforms differently to the general element $A$ of the algebra. This is because $U$ itself is an element of the ideal, and so transforms by

$$U \to QU. \tag{2.7a}$$

But (2.1) can be written as

$$AU \to (QAQ^{-1})(QU)$$

which implies that the general element $A$ must transform by the equivalence transformation

$$A \to QAQ^{-1}. \tag{2.7b}$$

Thus, although the idempotent is defined by (1.10) in terms of the elements of the algebra, it does not transform as an element itself. So the constituents of the ideal must be regarded as a distinct set of mathematical entities which are subject to the one-sided transformation law (2.1*a*).

A further problem is that, as in (2.7*a*), the transformed idempotent is $QU$: but for most transformations $Q$,

$$(QU)^2 \neq QU.$$

So the transformation does not transform an idempotent into an idempotent.

There is an alternative to this process of defining an ideal in terms of elements of the algebra, and then ascribing to it a transformation law different to that of the elements themselves. We can treat an idempotent (1.10) and the corresponding ideal (1.9) as elements of the algebra $R_{p,q}$, and consider gauge transformations which are local changes of representation by a position-dependent equivalence transformation of all elements $X$, including the spinors:

$$X \to Q(x)XQ^{-1}(x). \tag{2.8}$$

These transformations differ in several ways from Yang–Mills transformations, and it is the purpose of this paper to study these differences. One difference is that the ideal defining the spinor changes under the transformation (2.8), but this has the advantage that an idempotent transforms into an idempotent, since

$$(QUQ^{-1})^2 = QU^2Q^{-1} = QUQ^{-1}.$$

This ensures that an ideal transforms into an ideal. The change in the form of gauge transformation does not imply any change in the (left-sided) operation of the rotation or Lorentz group on a spinor, except that the representatives of these groups are themselves transformed by (2.8).

If $\phi$ and $\psi$ are algebraic spinors, the transition amplitude from $\psi$ to $\phi$ of an element $A$ is the scalar part of $\bar{\phi}A\psi$, denoted by $\langle\bar{\phi}A\psi\rangle_S$. Likewise, the free Lagrangian is defined as

$$\langle\bar{\psi}\mathrm{i}e^{\mu}(\partial_{\mu}\psi)\rangle_S \tag{2.9}$$

where all terms in the Lagrangian are viewed as being elements of the Clifford algebra, including the imaginary unit i. We have discussed elsewhere the existence within a Clifford algebra of an imaginary unit [3, 5]. To summarize, the algebra $R_{p,q}$, $p + q = n$, contains an element which can be identified as having properties equivalent to the imaginary unit provided that $(q - p) = 1 \bmod(4)$. In these algebras the imaginary unit is represented by the pseudoscalar, that is, the basis multivector which is the product of all $n$ basis vectors. The definition of a Lagrangian density of the form (2.9) depends not only on the existence of the

imaginary unit i but also on the operator $\Gamma$. Thus it can only be defined in a much more restricted class of algebras $R_{p,q}$ for which $(p-q) = 3 \bmod(4)$ and $q$ is even, $q \neq 0$ [3].

Modification of the gauge transformation on the spinors to (2.8) has implications when we consider invariance of the Lagrangian (2.9). It will not be invariant under the transformation (2.8) if we replace the partial derivative with the covariant derivative defined by (2.2). Invariance can be achieved if, instead of using (2.2), we redefine the covariant derivative as

$$D_\mu \psi = \partial_\mu \psi - [\Omega_\mu, \psi] \tag{2.10}$$

and still retain the same transformation (2.3) of $\Omega_\mu$. The definition (2.10) of the covariant derivative ensures that it transforms according to (2.8) like all the other elements of the algebra. Thus the Lagrangian transforms

$$\langle \bar{\psi} \mathrm{i} e^\mu (D_\mu \psi) \rangle_S \rightarrow \langle Q \bar{\psi} \mathrm{i} e^\mu (D_\mu \psi) Q^{-1} \rangle_S.$$

By using the cyclic property (1.22), this is easily shown to be an invariant quantity.

In the algebraic representation, the transformation of $\Omega_\mu$ is best thought of as consisting of two parts. This becomes explicit if we write it as:

$$\Omega_\mu \rightarrow Q(x)[\Omega_\mu + Q^{-1}(\partial_\mu Q)]Q^{-1}(x). \tag{2.11}$$

Here $\Omega_\mu$ as an element of $R_{p,q}$ undergoes a similarity transformation as all the other elements do. However, we require it additionally to be a potential with particular gauge transformation properties in accordance with the Yang–Mills formulation.

The invariant Lagrangian

$$\langle \bar{\psi} \mathrm{i} e^\mu (D_\mu \psi) \rangle_S \tag{2.12}$$

contains the usual spinor kinetic energy term and the usual interaction term with the vector boson fields, $-\langle \bar{\psi} \mathrm{i} e^\mu \Omega_\mu \psi \rangle_S$. However, it also contains the term

$$\langle \bar{\psi} \mathrm{i} e^\mu \psi \Omega_\mu \rangle_S \tag{2.13}$$

which is not of traditional bispinor density form. Other authors [6, 7] have considered terms such as these and have tried interpreting them in a variety of ways. For example, since they have $\Omega_\mu$ operating on $\psi$ on the right, these terms mix different minimal left ideals, so they have been given a family interpretation. We make a different proposal about these terms and show that they contribute to the usual interaction terms with the vector bosons. In the next section we will concentrate on general algebraic results which enable us to make this case.

## 3. General results for all algebras $R_{p,q}$

In (1.10) we defined a primitive idempotent as a product of $k$ simple idempotents, where $k$ is defined by the Radon–Hurwitz number. Each of the $k$ simple idempotents has the form

$$\tfrac{1}{2}(I + p_i) \qquad i = 1, 2, \ldots, k \tag{3.1}$$

with the projectors $p_i$ satisfying $p_i^2 = I$ and $[p_i, p_j] = 0$ for $i, j = 1, 2, \ldots k$. We can without loss of generality choose the projectors $\{p_i; i = 1, 2, \ldots, k\}$ to be basis multivectors in the algebra $R_{p,q}$ since, if they are not already multivectors, then the representation of the algebra can be changed so that they are.

The choice of positive sign in a simple idempotent is arbitrary and could be replaced by a minus sign. Thus we define a pair of possible simple idempotents $U_i^\pm$ as

$$U_i^\pm = \tfrac{1}{2}(I \pm p_i). \tag{3.2}$$

These two idempotents are mutually annihilating, that is,

$$U_i^+ U_i^- = U_i^- U_i^+ = 0. \tag{3.3}$$

We now collect together some results about the idempotents and the projectors, which we shall use to derive the main result of this section.

**Lemma 3.1.**

*(i)* $U_i^\pm p_j = p_j U_i^\pm$ *where* $i \neq j$.
*(ii)* $[U_i^\pm, U_j^\pm] = 0$ *for all* $i$ *and* $j$.
*(iii)* $U_i^\pm p_i = \pm p_i U_i^\pm$.
*(iv) If* $\alpha$ *is some multivector basis element of the algebra and* $\{\alpha, p_i\} = 0$, *then* $U_i^\pm \alpha = \alpha U_i^\mp$.
*(v) If* $\alpha$ *is some multivector basis element of the algebra and* $[\alpha, p_i] = 0$, *then* $U_i^\pm \alpha = \alpha U_i^\pm$.

**Proof.**

(i) This follows automatically from the definition of the primitive idempotent (1.10) in which the set of projectors $\{p_i\}$ must be a commuting set.
(ii) These results follow directly from (i).
(iii) This follows from the definitions of $U_i^\pm$ and the property $p_i^2 = I$.
(iv) If $\{\alpha, p_i\} = 0$, then it follows that $(I - p_i)\alpha = \alpha(I + p_i)$. Thus the result follows.
(v) The result follows in a similar way to (iv).

□

We are now able to prove the main result which will be needed in the next section.

**Theorem 3.2.** *Let* $U$ *be a primitive idempotent in* $R_{p,q}$ *which is a product of the* $k$ *simple idempotents* $U_i$ *which each have the form of one of the idempotent pair (3.2) and where* $k$ *is the Radon–Hurwitz number. If* $e_T$ *is any basis multivector of the algebra which is self conjugate, that is,* $e_T = e_T^\dagger$, *then*

$$U e_T U = \lambda U$$

*where* $\lambda$ *is either* $+1$, $-1$ *or* $0$.

**Proof.** In the proof we shall assume that the primitive idempotent $U$ is a product of simple idempotents each of the form $U_i^+$, that is,

$$U = U_1^+ U_2^+ \dots U_j^+ \dots U_k^+.$$

The proof proceeds in a similar way if $U$ is a product of idempotents each of the form $U_i^-$, or a mixture of $U_i^+$ and $U_i^-$.

Since $e_T$ is self-conjugate, it must by (1.13) square to $I$. The term $e_T$ must then be a member of one of two discrete sets:

(a) those in which the multivector is a product of elements of the set $\{I, p_i; i = 1, 2, \dots, k\}$;
(b) those in which the multivector is a product of elements of the set $\{I, p_i; i = 1, 2, \dots, k\}$ *together with* other multivector basis elements $\{\alpha_r\}$.

Thus the terms in the set (b) are all products of multivectors containing at least one factor $\alpha_r$ distinct from $\{p_i\}$. We note that the set (b) cannot contain *all possible* products of the elements $\{I, p_i, \alpha_r\}$ since some of these products will have square $-I$.

We shall establish the theorem in several stages. First consider those terms $e_T$ which belong to set (a):

(i) The term in (a) which is proportional to $I$ is automatically in the required format since $UIU = U^2 = U$.

(ii) Next consider terms in (a) which are proportional to a single projector $p_j$:
In the expression $Up_j = U_1^+U_2^+ \ldots U_j^+ \ldots U_k^+ p_j$, the projector can be commuted past each $U_i^+ \; i > j$, by lemma 3.1 (i). When $p_j$ meets $U_j^+$ it will be 'absorbed' using lemma 3.1 (iii). Thus

$$Up_j = U.$$

If $U_j^-$ appears in $U$ instead of $U_j^+$ then the overall result is only changed by an overall minus sign.

(iii) The process of 'absorption' can readily be extended to terms in set (a) which are a product of several projectors. So

$$U\prod_s p_s = U.$$

Again there may be a sign change in this result if $U$ is comprised of some simple idempotents of the form $U_i^-$.

Thus any multivector in the set (a) produces a term $\lambda U^2 = \lambda U$, where $\lambda = \pm 1$.

We now consider the terms in set (b).

(iv) Consider first the collection of terms in the set (b) which are proportional to a single multivector basis element $\alpha_r$. We recall that $\alpha_r$ is not equal to any of the elements in set (a) and it squares to give $I$. The set of projectors $\{p_i\}$ which are used to form a primitive idempotent is a *maximal* set of commuting elements with square $I$. Thus the element $\alpha_r$ must anti-commute with at least one of the projectors in the set $\{p_i\}$. Let the greatest value of the subscript $i (i = 1, 2, \ldots, k)$ for which $\alpha_r$ anti-commutes with $p_i$ be denoted by $i = j$.
In the expression $U\alpha_r = U_1^+U_2^+ \ldots U_j^+ \ldots U_k^+ \alpha_r$, the element $\alpha_r$ can be commuted past each $U_i^+ \; i > j$ without changing its form by lemma 3.1 (v). When $\alpha_r$ meets $U_j^+$, it can by lemma 3.1 (iv) be taken through $U_j^+$, but in doing so will convert it to $U_j^-$ giving

$$\begin{aligned} U\alpha_r &= U_1^+U_2^+ \ldots \alpha_r U_j^- U_{j+1}^+ \ldots U_k^+ \\ &= U_1^+U_2^+ \ldots \alpha_r U_{j+1}^+ \ldots U_k^+ U_j^- \end{aligned}$$

by using lemma 3.1 (ii). Thus

$$\begin{aligned} U\alpha_r U &= U_1^+U_2^+ \ldots \alpha_r U_{j+1}^+ \ldots U_k^+ U_j^- U_1^+U_2^+ \ldots U_j^+ \ldots U_k^+ \\ &= U_1^+U_2^+ \ldots \alpha_r U_{j+1}^+ \ldots U_k^+ U_j^- U_j^+ U_1^+U_2^+ \ldots U_k^+ \end{aligned} \tag{3.4}$$

making use of lemma 3.1 (ii) to rearrange the order of the simple idempotents.
By using (3.3), the pair of idempotents $U_j^-U_j^+$ in the centre of this expression annihilate one another. Thus the terms which are of the form $U\alpha_r U$ are zero.

(v) The reduction of terms to zero also applies to the other terms in set (b), that is, those containing a product of a number of terms in set $\{I, p_i; i = 1, 2, \ldots, k\}$ together with some of the elements $\{\alpha_r\}$. The argument is essentially a combination of the arguments given above: any $p_i$ is absorbed, but commutation of some $\{\alpha_r\}$ through $U$ leads to an expression similar to (3.4), in which at least one $U_j^+$ on the left is replaced by $U_j^-$, ensuring that the element is zero. It is at this stage that multiplicative independence is important.

If $U$ contains $U_j^-$ rather than $U_j^+$ then the argument is similar since the anti-commutation of $\alpha_r$ will convert $U_j^-$ into $U_j^+$ which will then ensure annihilation. □

**Corollary 3.2.** *In algebras $R_{p,q}$ for which $(q-p) = 1 \bmod(4)$ then, with no restrictions on $e_T$,*

$$Ue_TU = \zeta U$$

*where $\zeta$ is either $\pm\mathrm{i}, \pm 1$ or 0.*

**Proof.** As we mention in section 2, when $(q-p) = 1 \bmod(4)$, the pseudoscalar of the algebra can be considered to be the unit imaginary i. The basis multivectors of the algebra can be divided into two sets $\{e_T\}$ and $\{\mathrm{i}e_T\}$. The elements in the first set can be divided into two: those which square to $I$ and are denoted by $\{f_T\}$; and those which square to $-I$ denoted by $\{g_T\}$. Then the complete set of basis multivectors which square to $I$ is $\{f_T, \mathrm{i}g_T\}$. Theorem 3.2 will apply directly to any basis multivector from the set $\{f_T, \mathrm{i}g_T\}$. However, none of the arguments in the proof of the theorem would be affected if we were to consider $e_T$ instead as a member of $\{\mathrm{i}f_T, g_T\}$. The only difference would be in the final outcome; that is, under these circumstances,

$$Ue_TU = \pm\mathrm{i}U$$

where i should be regarded as the pseudoscalar of $R_{p,q}$. This is quite consistent with referring to it as the imaginary unit since it satisfies all the required properties. □

The proof of theorem 3.2 and its corollary demonstrates that a term of the form $Ue_TU$ with $e_T$ self-conjugate is only non-zero when $e_T$ is proportional to products of elements of the set $\{I, p_i, i = 1, 2, \ldots, k\}$; in these terms the projectors $p_i$ are 'absorbed' into $U$ with non-zero coefficient. Thus, all terms $Ue_TU$ reduce to $\zeta U^2 = \zeta U$, where $\zeta = \pm\mathrm{i}, \pm 1$ or 0.

In appendix A, examples 1 and 2 demonstrate the use of theorem 3.2 and corollary 3.2 to evaluate terms of the form $Ue_TU$ in the algebras $R_{3,0}$ and $R_{2,3}$.

## 4. Consideration of the additional interaction terms

In section 2 we remarked that the extra terms produced by the two-sided gauge transformation of the spinor were not of the traditional bispinor density form. We shall now use the results of section 3 to reconsider these terms.

By taking the definition (2.6) of a spinor as an element of a minimal left ideal of the Clifford algebra $R_{p,q}$, the bar conjugate spinor defined as in (1.16) becomes

$$\bar{\psi} = U^\dagger \Gamma A^R. \tag{4.1}$$

Since the projectors $p_i$ in the simple idempotents comprising $U$ are such that $p_i^2 = I$, they are by (1.13) self-Hermitian. By applying lemma 3.1 (ii) we can reverse the order of the simple idempotents in $U^\dagger$ so that $U^\dagger = U$ and (4.1) becomes

$$\bar{\psi} = U\Gamma A^R.$$

Then the 'new' interaction term (2.13) can be written as

$$\langle \bar{\psi}\mathrm{i}e^\mu \psi \Omega_\mu \rangle_S = \langle U\Gamma A^R \mathrm{i}e^\mu AU\Omega_\mu \rangle_S \tag{4.2}$$

$$= \langle \Gamma A^R \mathrm{i}e^\mu AU\Omega_\mu U \rangle_S. \tag{4.3}$$

The connection $\Omega_\mu$ in (4.3) can be expressed as a sum of terms each of which has the form of an $x$-dependent real field, which is a Clifford scalar, multiplied by a multivector basis element of $R_{p,q}$. Let a typical term in $\Omega_\mu$ be denoted by $W_\mu^T e_T$, where $W_\mu^T$ is a Clifford scalar. We shall illustrate the reduction of (4.3) by considering this typical term only. Given that the Lagrangian has to be defined using an imaginary unit i and the operator $\Gamma$, a Lagrangian field

theory based on (4.2) can only be defined for algebras $R_{p,q}$ for which $(p-q) = 3 \bmod(4)$ and $q$ is even. This is a special case of the algebras to which corollary 3.2 applies. Thus, applying corollary 3.2 to the expression $U\Omega_\mu U$ within (4.3), it follows that the contribution from the term $W_\mu^T e_T$, if it is not zero, is of the form

$$\begin{aligned}\zeta\langle\Gamma A^R \mathrm{i}e^\mu A W_\mu^T I U\rangle_S &= \zeta\langle\Gamma A^R \mathrm{i}e^\mu A W_\mu^T I U^2\rangle_S \\ &= \zeta\langle\Gamma A^R \mathrm{i}e^\mu W_\mu^T A U^2\rangle_S \\ &= \zeta\langle U\Gamma A^R \mathrm{i}e^\mu W_\mu^T A U\rangle_S \\ &= \zeta\langle\bar\psi \mathrm{i}e^\mu W_\mu^T I\psi\rangle_S \end{aligned} \tag{4.4}$$

where $\zeta$ is $\pm1$ or $\pm\mathrm{i}$, that is, plus or minus the scalar or pseudoscalar of the algebra. The only terms which remain from (4.3) will be a sum of terms of the form (4.4). These all have the form of standard interactions since they all have bispinor density form. In particular the 'real' terms of type (4.4) have the form of a standard $U(1)$ interaction

$$-\langle\bar\psi e^\mu W_\mu^T I\psi\rangle_S. \tag{4.5}$$

Given a symmetrized Lagrangian with kinetic terms for both $\psi$ and $\bar\psi$, the 'imaginary' terms of type (4.4) will vanish.

The appearance of the extra terms (4.5), which have the form of a $U(1)$ interaction, is a completely new feature of gauge theory. However, if we treat the spinors as normal elements of a Clifford algebra, we have to accept the existence of these new interaction terms and explain their meaning. We have proposed an answer which applies to all possible interaction terms. In appendix B, we have considered what effect the inclusion of the new interaction terms (4.5) has on the Glashow description of electroweak interactions of leptons [8]. If we make a particular choice of idempotent, a remarkable result follows: we need only include in $\Omega_\mu$ those parts of the Glashow interactions incorporating $SU(2)$ generators. Then the extra term arising in our new covariant derivative gives precisely the additional term in the observed interactions. So our gauge transformations are simpler than those assumed in the standard theory, but give rise to the usual leptonic electroweak interactions.

## 5. A new definition of spinors

Although we have already established our required outcome in section 4, we shall investigate further the results of section 3. We shall show that it is possible to use them to define an alternative but equivalent Clifford algebraic definition of spinors. The alternative definition is based on the observation that the idempotents $U_i^\pm$ can each be factorized into the product of a pair of 'conjugate' nilpotents, as demonstrated in the following theorem.

**Theorem 5.1.** *If $U_i^\pm$ is written as*

$$U_i^\pm = \tfrac{1}{4}(H_i \pm A_i)(H_i \mp A_i) \tag{5.1a}$$

*with nilpotent factors satisfying*

$$(H_i + A_i)^2 = (H_i - A_i)^2 = 0 \tag{5.1b}$$

*then $H_i$ and $A_i$ are such that*

$$H_i^2 = -A_i^2 = I \qquad \{H_i, A_i\} = 0 \qquad p_i = A_i H_i = -H_i A_i. \tag{5.2}$$

**Proof.** The proposed factorization of $U_i^\pm$ implies that

$$\begin{aligned} U_i^+ &\equiv \tfrac{1}{2}(I + p_i) = \tfrac{1}{4}(H_i^2 - A_i^2 + [A_i, H_i]) \\ U_i^- &\equiv \tfrac{1}{2}(I - p_i) = \tfrac{1}{4}(H_i^2 - A_i^2 - [A_i, H_i]). \end{aligned}$$

Adding and subtracting these equations give

$$I = \tfrac{1}{2}(H_i^2 - A_i^2) \tag{5.3a}$$

$$p_i = \tfrac{1}{2}[A_i, H_i]. \tag{5.3b}$$

From conditions (5.2), we can deduce that

$$H_i^2 + A_i^2 + \{A_i, H_i\} = 0$$
$$H_i^2 + A_i^2 - \{A_i, H_i\} = 0.$$

Adding and subtracting these equations give

$$H_i^2 + A_i^2 = 0 \tag{5.3c}$$

$$\{H_i, A_i\} = 0. \tag{5.3d}$$

The various equations (5.3) then yield the required results. □

In appendix A, we demonstrate in example 3 how a familiar idempotent of the Pauli algebra $R_{3,0}$ can be factorized into two equally familiar conjugate nilpotents. This is the simplest and most fundamental factorization into nilpotents. In example 4 we consider the factorization for the algebra $R_{2,3}$.

We have already noted in section 3 that we can without loss of generality choose the projectors $\{p_i; i = 1, 2, \ldots, k\}$ to be basis multivectors in the algebra $R_{p,q}$. For each algebra $R_{p,q}$ there are always at least $2k$ independent multivectors which square to $I$. Thus having chosen $p_i$ as basis multivectors we can choose the form of the factorization in theorem 5.1 so the elements $\{H_i; i = 1, 2, \ldots, k\}$ are also basis multivectors and hence are invertible. Having chosen $p_i$ and $H_i$ in this way the terms $A_i$ in the factorization are then defined as

$$A_i = H_i^{-1} p_i \qquad i = 1, 2, \ldots, k.$$

This implies that all of the $A_i$ are also basis multivectors. From (1.14) we can then deduce the following hermiticity properties for $p_i$, $H_i$ and $A_i$:

$$p_i^\dagger = p_i \qquad H_i^\dagger = H_i \qquad A_i^\dagger = -A_i \qquad (i = 1, 2, \ldots, k). \tag{5.4}$$

This explains our use of the term 'conjugate' when applied to the pair of terms which comprise the factorization of $U_i^\pm$. Each idempotent is factorized into a pair of nilpotents, one of which is the Hermitian conjugate of the other. We denote the nilpotents by

$$N_i = \tfrac{1}{2}(H_i + A_i) \qquad i = 1, 2, \ldots, k. \tag{5.5a}$$

Then

$$U_i^+ = N_i N_i^\dagger \qquad \text{and} \qquad U_i^- = N_i^\dagger N_i \tag{5.6}$$

where

$$N_i^\dagger = \tfrac{1}{2}(H_i - A_i). \tag{5.5b}$$

The results of theorem 3.2 and its corollary lead us to similar results in which the idempotents are replaced by the nilpotents. But first we note some identities which relate the nilpotents $N_i$ and $N_i^\dagger$ to the idempotent $U_i^+$:

$$U_i^+ = N_i H_i \tag{5.7a}$$

$$U_i^+ = H_i N_i^\dagger. \tag{5.7b}$$

The identities (5.7) follow directly from the definitions (3.2), (5.5*a*) and (5.5*b*) for $U_i^+$, $N_i$ and $N_i^\dagger$ respectively and using (5.1*b*) for $p_i$.

By using (5.7) the primitive idempotent $U$ in theorem 3.2 can be written as

$$U = N_1 H_1 N_2 H_2 \ldots N_k H_k \tag{5.8a}$$

or, since the simple idempotents in $U$ commute and so can appear in any order, it can also be written as

$$U = H_k N_k^\dagger \ldots H_2 N_2^\dagger H_1 N_1^\dagger. \tag{5.8b}$$

Thus the expression $U e_T U$ to which theorem 3.2 applies is equivalent to

$$N_1 H_1 N_2 H_2 \ldots N_k H_k e_T H_k N_k^\dagger \ldots H_2 N_2^\dagger H_1 N_1^\dagger. \tag{5.9}$$

The Hermitian term $H_i$ in this expression is a basis multivector and thus will either commute or anti-commute with the factors $A_j$ and $H_j$ comprising $N_j$ and $N_j^\dagger$. Thus

$$\begin{array}{lll} H_i N_j = N_j H_i \text{ and } H_i N_j^\dagger = N_j^\dagger H_i & & \text{if } [A_j, H_i] = [H_j, H_i] = 0,\ i \neq j \\ \quad = -N_j H_i & = -N_j^\dagger H_i & \text{if } \{A_j, H_i\} = \{H_j, H_i\} = 0,\ i \neq j \\ \quad = N_j^\dagger H_i & = N_j H_i & \text{if } \{A_j, H_i\} = [H_j, H_i] = 0,\ i \neq j \\ \quad = -N_j^\dagger H_i & = -N_j H_i & \text{if } [A_j, H_i] = \{H_j, H_i\} = 0,\ i \neq j. \end{array} \tag{5.10}$$

This means that all of the $H_i$ on the left and right of $e_T$ in (5.9) can be brought through the $N_i^\dagger$ and $N_i$ terms and the only effect will be to change some $N_i$ to $N_i^\dagger$ and vice versa and to introduce a possible overall sign change. It is important to note that if some $H_i$ on the right when brought through a particular $N_j^\dagger$ converts it to $N_j$, then the same $H_i$ on the left when brought through $N_j$ will convert it to $N_j^\dagger$. Thus, by using (5.10) repeatedly, we can show that (5.9) can be written as

$$N H_k \ldots H_2 H_1 e_T H_1 H_2 \ldots H_k N^\dagger \tag{5.11}$$

where

$$N = \prod_{i=1}^{k} (\pm N_i^{(\dagger)}). \tag{5.12}$$

The sign in each term in the product in (5.12) and also whether or not the Hermitian conjugate $N_i^\dagger$ (as denoted by the $\dagger$ sign appearing in brackets) is used depends on the outcome of bringing each of the $H_j$, $j = 1, 2, \ldots k$, through each of the $N_j^\dagger$, $j = 1, 2, \ldots k$, as determined by (5.10).

Since each $H_i$, $i = 1, 2, \ldots k$, is a basis multivector, it will either commute or anti-commute with the basis multivector $e_T$. Thus, since $H_i^2 = I$, (5.11) reduces to

$$\pm N e_T N^\dagger. \tag{5.13a}$$

In the simple case when $e_T = I$, then (5.11) reduces to

$$N N^\dagger. \tag{5.13b}$$

Hence using (5.13*a*) and (5.13*b*) we find, respectively, that

$$U e_T U \equiv \pm N e_T N^\dagger \tag{5.14a}$$

and

$$U = U^2 = N N^\dagger \tag{5.14b}$$

thus the statement of theorem 3.2 is equivalent to

$$N e_T N^\dagger = \lambda N N^\dagger \tag{5.15}$$

where $\lambda = \pm 1$ or 0. This process of absorption by nilpotents of the algebras $R_{3,0}$ and $R_{2,3}$ is discussed in examples 3 and 4 of appendix A.

Given the 'equivalent' status of the idempotent $U$ and the nilpotent $N$ in respect of theorem 3.2, it is interesting to speculate whether there is an equivalent Clifford algebraic definition of spinors based on nilpotents as opposed to idempotents. We complete this section by demonstrating this fact.

The alternative definition of a spinor takes as its starting point the product of nilpotents $N$. A spinor $\chi$ is defined to be an element of the set $\{CN, C \in R_{p,q}\}$. The bar conjugate spinor $\bar{\chi}$ is defined as in (1.16) and is given by

$$\bar{\chi} = N^{\dagger}\Gamma C^{R}. \tag{5.16}$$

It is important to show that this new definition of a spinor is equivalent, when evaluating standard interaction terms, to the definition in section 1 using the primitive idempotents. To demonstrate this, we consider a bispinor density term using the new definition of spinors and relate it to the definition in terms of idempotents using (5.15) which connects $U$ to $N$:

$$\begin{aligned}\langle\bar{\chi}B\chi\rangle_S &= \langle N^{\dagger}\Gamma C^{R}BCN\rangle_S \\ &= \langle\Gamma C^{R}BCNN^{\dagger}\rangle_S \\ &= \langle\Gamma C^{R}BCU\rangle_S \\ &= \langle\Gamma C^{R}BCU^{2}\rangle_S \\ &= \langle U\Gamma C^{R}BCU\rangle_S. \end{aligned}\tag{5.17}$$

The expression (5.17) is identical to, and contains the same information as, the bispinor density defined using the spinor $\psi$ and its bar conjugate $\bar{\psi}$, which are

$$\psi = CU \tag{5.18a}$$

$$\bar{\psi} = U\Gamma C^{R}. \tag{5.18b}$$

The alternative spinor $\chi$ has the same properties as $\psi$ in respect of the additional terms in the Lagrangian of the form

$$\begin{aligned}\langle\bar{\chi}B\chi e_T\rangle_S &= \langle N^{\dagger}\Gamma C^{R}BCNe_T\rangle_S \\ &= \langle\Gamma C^{R}B\,CNe_TN^{\dagger}\rangle_S.\end{aligned}$$

They too will reduce to standard $U(1)$ interaction terms by virtue of the result (5.15).

## 6. Discussion

In previous work [9–12], we have developed spin gauge models of families of elementary particles and their interactions, using the standard Clifford algebraic definition of spinors as left ideals of an algebra, based on idempotents. We have also assumed that the interactions were generated by gauge transformations of Yang–Mills form. We have been aware of possible differences between the Clifford algebra approach and the standard theory based on Hilbert space concepts.

More recently, we have developed and extended work by Li [13], seeking to define all quantities and operations in spin gauge theory models on a proper algebraic basis, spinors being identified with standard minimal left ideals [3]. This study has led us to consider gauge theories in which algebraic spinors are transformed in the same way as general elements of an algebra, by two-sided equivalence transformations. To us, this seems the simplest and most natural assumption for transformations which correspond to no physically observable effects. However, two-sided gauge transformations of spinors imply that the corresponding Lagrangian interaction terms contain the *commutator* $[\Omega_\mu, \psi]$ of the interaction potential with the spinor, rather than the usual product $\Omega_\mu\psi$ of these terms. So a new type of interaction term arises, in

which the interaction potential appears on the 'wrong side' of the spinor. The solution of one problem gives rise to a further problem!

It is important to note that the two-sided gauge transformations are impossible for standard column spinors. The new type of gauge transformation is thus only applicable to algebraic spinors. In the algebraic formulation, the Lagrangian is the scalar part of an element of an algebra $R_{p,q}$, with $p - q = 3$ (not 4) and $q$ even, $q \neq 0$, that is a 'physical' algebra. Our new interaction term occurs, within the scalar part, sandwiched between the spinor idempotents both on the left and on the right. We have been able to interpret and evaluate this term using the result established in corollary 3.2, that *any* element of the algebra lying between two primitive idempotent operators reduces to a multiple (sometimes zero) of the Clifford scalar. The element is thus 'absorbed' by the idempotents. So any possible interaction term 'on the wrong side' is either zero or is equivalent to a potential generated by a $U(1)$ gauge transformation.

We have shown that the result of theorem 3.2 can be expressed in an alternative but equivalent way using conjugate nilpotent pairs which arise in the factorization of each of the idempotents in the primitive idempotent. This leads to an alternative but equivalent definition of Clifford algebraic spinors.

In appendix B, we have proposed an alternative account of the electroweak interactions of leptons [8]. We find that if the usual $SU(2)$ interaction terms are generated by gauge transformations, a suitable choice of idempotent or nilpotent factors ensures that the 'extra' interaction potential is precisely equal to the additional $U(1)$ interaction term, with the correct coupling constant. So the helicity-minus electroweak interactions arise from a simpler gauge transformation than in standard theory. Our new form of gauge theory has thus led to an unexpected simplification in the basic electroweak gauge transformations for leptons. This simplification is very welcome, but these ideas need to be applied in a wider context, to cover both strong and gravitational interactions. We have already developed a spin gauge model incorporating all particle interactions [12], but this now needs to be modified to incorporate the new type of interaction term. The incorporation of gravitational interactions requires space-time to be curved, but this should not affect the introduction of two-sided transformations and nilpotents described in this paper. Our interpretation of fermion mass terms as 'frame field interactions' must also be studied in this new context, but our first impressions are that no insuperable problems will arise, because of the comprehensive nature of theorem 3.3 and corollary 3.3, and of our nilpotent factorization.

## Appendix A

In the first two examples in this appendix, we illustrate the process of 'absorption' and 'annihilation' of typical elements of an algebra by an idempotent. Our third and fourth examples demonstrate the factorization of minimal idempotents into conjugate pairs of nilpotent factors.

**Example 1.** The simplest algebra for which the absorption property is nontrivial is the physically important Pauli algebra $R_{3,0}$. An orthonormal set of basis vectors $\{e_r; r = 0, 1, 2\}$ of the algebra satisfy the anti-commutation relations

$$\{e_r, e_s\} = 2I\delta_{rs} \tag{A.1}$$

where $I$ is the unit of the algebra. The well known Pauli matrices $\{\rho_s; s = r + 1\}$ form a representation of these basis vectors. The pseudoscalar $\omega = e_0e_1e_2$ satisfies $\omega^2 = -I$ and commutes with every element of the algebra; so it behaves in every respect like the imaginary unit, and we shall use the notation '$i$' for $\omega$. Note that $p = 3$ and $q = 0$ satisfy

$q - p = 1 \bmod(4)$, the condition for the pseudoscalar to be taken as the imaginary unit. The bivectors of the algebra can be written as $\{\mathrm{i}e_r\}$.

The maximum number of commuting projectors with square $I$ is

$$k = q - r_{q-p} = 0 - r_{-3} = 4 - r_5 = 1$$

and we can choose this projector as $p = e_0$, so that the primitive idempotent can be chosen to be

$$U = \tfrac{1}{2}(I + e_0).$$

We now show that all elements of the Pauli algebra are 'absorbed' or 'annihilated' when an idempotent factor $U$ is adjacent to the element on the right and on the left, in accord with corollary 3.2. Since the pseudoscalar i commutes with the whole algebra, we need only show that the three basis vectors are absorbed:

$$(I + e_0)e_0(I + e_0) = (I + e_0)^2 = 2(I + e_0)$$
$$(I + e_0)e_1(I + e_0) = e_1(I - e_0)(I + e_0) = 0.$$

We note that the projector $e_0$ is absorbed with a non-zero coefficient, whereas $e_1$ (and likewise $e_2$) gives zero on absorption, through the idempotent condition.

**Example 2.** Our second example uses the Clifford algebra appropriate to Dirac's theory of the electron. The real algebra $R_{1,3}$ is not the appropriate Clifford algebra, since it does not contain an element with the properties of the imaginary unit. The appropriate algebra is $R_{2,3}$, which satisfies the condition $q - p = 1$ and whose pseudoscalar element can therefore be used as the imaginary unit i. An orthonormal vector basis $\{e_r; r = 0, 1, 2, 3, 4\}$ has anti-commuting elements satisfying

$$e_0^2 = e_1^2 = -e_2^2 = -e_3^2 = -e_4^2 = I. \tag{A.2}$$

The basis elements of the algebra $R_{2,3}$ are all products of these five basis vectors. We can alternatively derive all of the basis elements as products of five multiplicatively independent basis elements of $R_{2,3}$, where multiplicative independence is defined in section 1. For example, we can choose to define all elements as products of the five elements {i and $e_r; r = 1, 2, 3, 4$}.

The maximum number of commuting projectors is

$$k = q - r_1 = 3 - 1 = 2$$

and they can be chosen to be $e_1e_4 \equiv e_{14}$ and $\mathrm{i}e_2e_3 \equiv \mathrm{i}e_{23}$. The four possible primitive idempotents are then

$$\tfrac{1}{4}(I \pm e_{14})(I \pm \mathrm{i}e_{23}).$$

We shall use the idempotent

$$U = \tfrac{1}{4}(I + e_{14})(I + \mathrm{i}e_{23})$$

in this example.

By corollary 3.2, the basis elements of the algebra divide into two sets of terms: those that are 'absorbed' when they occur sandwiched between two factors $U$ and those that are annihilated. We shall identify these two sets of terms for this example, which are analogous to the two sets (a) and (b) in the proof of theorem 3.2. However, in this case, since the conditions of corollary 3.2 hold, the sets of type (a) and (b) are not restricted only to terms which square to $I$; the totality of basis elements divides into these two sets.

The chosen projector set is $\{e_{14}, \mathrm{i}e_{23}\}$. Either of these two elements is 'absorbed' when placed between factors $U$. For example, taking $e_T = e_{14}$,

$$\begin{aligned} Ue_{14}U &= \tfrac{1}{4}(I + e_{14})(I + \mathrm{i}e_{23})e_{14}U \\ &= \tfrac{1}{4}(I + e_{14})(I + \mathrm{i}e_{23})U = U^2. \end{aligned}$$

Any element which is a combination of these two elements and the imaginary unit i will also be absorbed in a similar way with a non-zero (complex) coefficient. Thus any linear combination of products of the elements $\{I, e_{14}, \mathrm{i}e_{23}, \mathrm{i}\}$ will also be absorbed.

We have noted that all the basis elements of $R_{2,3}$ can be derived from five multiplicatively independent basis elements. To identify sets analogous to types (a) and (b), we choose five suitable multiplicatively independent elements. We can select the projectors $e_{14}$ and $\mathrm{i}e_{23}$ as two of the five basis elements, and the pseudoscalar i can be used as a third independent element. The two other elements can be chosen to be, for example, $e_1$ and $e_{12}$. Then a suitable set of five elements from which all the basis elements can be derived is $\{e_{14}, \mathrm{i}e_{23}, \mathrm{i}, e_1, e_{12}\}$. If we consider the result of placing each of these five elements between idempotents, then we can derive the behaviour of all basis elements. The first three of these five forms set (a) and basis elements formed exclusively from these three have been considered above: they are absorbed with non-zero coefficients.

We now consider the remaining basis elements, each of which must contain at least one of $e_1$ and $e_{12}$ as a factor. All of these terms comprise set (b), that is, they get annihilated when placed between two nilpotents, since $e_1$ and $e_{12}$ get annihilated. To illustrate the process, the element we consider is $e_T = e_4 = -(e_{14})e_1$, containing the projector $e_{14}$ and the 'other' element $e_1$:

$$\begin{aligned} Ue_4U &= -\tfrac{1}{4}(I + e_{14})(I + \mathrm{i}e_{23})e_{14}e_1U \\ &= -\tfrac{1}{4}(I + e_{14})e_{14}e_1(I + \mathrm{i}e_{23})U \\ &= -\tfrac{1}{4}(I + e_{14})e_1(I + \mathrm{i}e_{23})U \\ &= -\tfrac{1}{4}e_1(I - e_{14})(I + \mathrm{i}e_{23})\tfrac{1}{4}(I + e_{14})(I + \mathrm{i}e_{23}) \\ &= -\tfrac{1}{8}e_1(I - e_{14})(I + e_{14})(I + \mathrm{i}e_{23}) = 0 \end{aligned}$$

using the idempotent property. We see that commutation of the factor $e_1$ has caused the crucial change of sign in $(I + e_{14})$ in this calculation, compared with the calculation above with $e_T = e_{14}$.

**Example 3.** In example 1, we used an idempotent of the Pauli algebra to exemplify the process of absorption. We now show how the idempotent can be factorized into a conjugate pair of nilpotent factors.

The nilpotent factors are of the form

$$N = \tfrac{1}{2}(H + A) \qquad N^\dagger = \tfrac{1}{2}(H - A) \tag{A.3}$$

and we take $H = e_1$ and $A = \mathrm{i}e_2$. Then the conditions of theorem 5.1 are satisfied, ensuring that

$$N^2 = N^{\dagger 2} = 0$$

and that

$$NN^\dagger = \tfrac{1}{4}(e_1 + \mathrm{i}e_2)(e_1 - \mathrm{i}e_2) = \tfrac{1}{2}(I + e_0) = U.$$

It is easy to see that $Ne_3N^\dagger = -NN^\dagger$, and that $Ne_1N^\dagger = Ne_2N^\dagger = 0$.

**Example 4.** As a second example of factorization into nilpotents, we use the idempotent

$$U = \tfrac{1}{4}(I + e_{14})(I + \mathrm{i}e_{23})$$

of example 2. The two simple idempotent factors of $U$ have nilpotent factors

$$\tfrac{1}{2}(I + e_{14}) = \tfrac{1}{4}(e_1 - e_4)(e_1 + e_4) \tag{A.4a}$$

$$\tfrac{1}{2}(I + \mathrm{i}e_{23}) = \tfrac{1}{4}(\mathrm{i}e_2 - e_3)(\mathrm{i}e_2 + e_3). \tag{A.4b}$$

If we define

$$N = \tfrac{1}{4}(e_1 - e_4)(\mathrm{i}e_2 - e_3) \tag{A.5}$$

and

$$N^\dagger = \tfrac{1}{4}(\mathrm{i}e_2 + e_3)(e_1 + e_4) \tag{A.6}$$

then anti-commutation of the brackets in $N$ and $N^\dagger$ ensures that

$$N^2 = N^{\dagger 2} = 0$$

while

$$NN^\dagger = \tfrac{1}{4}(I + e_{14})(I + \mathrm{i}e_{23}).$$

So $N$ and $N^\dagger$ form a conjugate pair of nilpotent factors of the idempotent $U$. It is easy to check that the elements $e_{14}$ and $e_4$ considered in example 2 are absorbed in a similar way by the nilpotent pair:

$$Ne_{14}N^\dagger = NN^\dagger \qquad Ne_4N^\dagger = 0.$$

## Appendix B

As we have mentioned in the main text, we have based a model [9] of the electroweak interactions on the Clifford algebra $R_{1,6}$. This algebra satisfies all of the conditions which are needed in order to formulate our type of gauge theory [5]. However, in this work we used one-sided gauge transformations of spinors, so that the new type of interaction terms did not occur. In this appendix, we examine what changes need to be made to the gauge transformations if we use two-sided gauge transformations of the form (2.8) on algebraic spinors, which are elements of minimal left ideals, in order to generate the correct electroweak interaction terms for leptons.

The vector basis $\{e_r; r = 0, 1, \ldots, 6\}$ of the algebra $R_{1,6}$ has

$$e_0^2 = I \qquad e_r^2 = -I \qquad (r = 1, 2, \ldots 6)$$

and the dimensions (0, 1, 2, 3) represent spacetime. Since $q - p = 6 - 1 = 5 = 1 \bmod(4)$, the pseudoscalar $\omega$ of the algebra can be identified with the imaginary unit, and corollary 3.2 applies. The maximum number of commuting projectors is

$$k = q - r_5 = 6 - 3 = 3$$

and we can take the three projectors as, for example,

$$\{e_{03}, \mathrm{i}e_{0123} \equiv \mathrm{i}\eta, \mathrm{i}e_{45} \equiv \rho_3\}. \tag{B.1}$$

The symbol $\eta$ in (B.1) is the spacetime pseudoscalar, and $\rho_3$ is one of the triple

$$\{\rho_1 = \mathrm{i}e_{56}, \rho_2 = \mathrm{i}e_{64}, \rho_3 = \mathrm{i}e_{45}\} \tag{B.2}$$

which satisfy the fundamental relations of the Pauli algebra. In our model, these elements are proportional to generators of the $SU(2)$ gauge transformations. From the three projectors (B.1)

we can define eight different primitive idempotents, each dependent on sixteen real parameters. We can use any one of these idempotents to define the spinor corresponding (in the first family of particles) to the electron and its neutrino. We find that the choice

$$U = \tfrac{1}{8}(I + e_{03})(I - \mathrm{i}\eta)(I - \rho_3) \equiv U_1^+ U_2^+ U_3^+ \tag{B.3}$$

is particularly interesting. The spacetime helicity projection operators are

$$h_\pm = \tfrac{1}{2}(I \pm \mathrm{i}\eta). \tag{B.4}$$

We note that one of these occurs in $U$, introducing a helicity asymmetry in the spinor. In our model, the electroweak interaction terms for leptons, placed in the 'usual' position in the Lagrangian, are

$$-\langle\bar{\psi}\mathrm{i}e^\mu\Omega_\mu\psi\rangle_S = -\left\langle\bar{\psi}\mathrm{i}e^\mu\left[(\mathrm{i}g/2)\sum_{r=1}^{3}h_+\rho_r W_{r\mu} + (\mathrm{i}g'/2)(h_-\rho_3 + I)W_{4\mu}\right]\psi\right\rangle_S \tag{B.5}$$

where $\{W_{r\mu}; r = 1, 2, 3, 4\}$ are the boson potentials and $g$ and $g'$ are the $SU(2)$ and $U(1)$ coupling constants. These terms, taken to be normal interaction terms with idempotent spinors, have been shown to generate the usual interactions for leptons [9]. The helicity projection operators have to be introduced because we include both right-handed and left-handed neutrino spinor states. The *helicity-dependent* part of this interaction involves

$$\Omega_{h\mu} = (\mathrm{i}g/2)\sum_{r=1}^{3}h_+\rho_r W_{r\mu} + (\mathrm{i}g'/2)h_-\rho_3 W_{4\mu}. \tag{B.6}$$

In the context of this paper using two-sided gauge transformations, the helicity dependent terms (B.6) may generate new terms equivalent to a $U(1)$ interaction of the form (4.2). To consider these new terms explicitly, we must evaluate the term (4.2) which contains

$$U\Omega_{h\mu}U. \tag{B.7}$$

First note that the helicity operators (B.4) commute with the factor $(I - \rho_3)$ in $U$. So from the first term in (B.6), the contribution to (B.7) contains the factors

$$(I - \mathrm{i}\eta)h_+ = 0.$$

So there is no contribution to the new interaction term (4.2) arising from the positive helicity term in (B.6). However, the negative helicity term in (B.6) does give a contribution, since

$$\tfrac{1}{2}(I - \mathrm{i}\eta)h_- = \tfrac{1}{2}(I - \mathrm{i}\eta).$$

Since the factor $\rho_3$ is absorbed through

$$(I - \rho_3)\rho_3 = -(I - \rho_3)$$

the extra interaction term arising from the second term in (B.6) contains

$$(\mathrm{i}g'/2)Uh_-\rho_3 W_{4\mu}U \to -(\mathrm{i}g'/2)IW_{4\mu}U^2. \tag{B.8}$$

The contribution of this term to the Lagrangian through (4.7) is therefore precisely the same as that normally derived from the *helicity-independent* $U(1)$ interaction term in (B.5). So this term does not have to be generated by a separate term in the gauge transformation. The surprising result is that the *full* gauge transformation needed to derive the observed lepton interactions is that normally required to generate only the helicity-dependent terms (B.6). The rather strange helicity-minus combination of terms observed in nature therefore arises from a simple gauge transformation.

We can alternatively adopt the new definition of spinors as described in section 5. We factorize each $U_i^+$, $i = 1, 2, 3$ according to (5.6) as follows,

$$\begin{aligned} U_1^+ &= \tfrac{1}{4}(e_0 - e_3)(e_0 + e_3) \equiv N_1 N_1^\dagger \\ U_2^+ &= \tfrac{1}{4}(e_0 + \mathrm{i}e_{123})(e_0 - \mathrm{i}e_{123}) \equiv N_2 N_2^\dagger \\ U_3^+ &= \tfrac{1}{4}(\mathrm{i}e_4 + e_5)(\mathrm{i}e_4 - e_5) \equiv N_3 N_3^\dagger. \end{aligned} \tag{B.9}$$

The nilpotents $H_i$, $i = 1, 2, 3$, in the factorization are $H_1 = H_2 = e_0$ and $H_3 = \mathrm{i}e_4$. Thus, from (5.8*a*) and (5.8*b*) the idempotent $U$ can be respectively written as

$$U = \tfrac{1}{2}(e_0 - e_3)e_0\tfrac{1}{2}(e_0 + \mathrm{i}e_{123})e_0\tfrac{1}{2}(\mathrm{i}e_4 + e_5)\mathrm{i}e_4 \tag{B.10$a$}$$

$$U = \tfrac{1}{2}\mathrm{i}e_4(\mathrm{i}e_4 - e_5)\tfrac{1}{2}e_0(e_0 - \mathrm{i}e_{123})\tfrac{1}{2}e_0(e_0 + e_3). \tag{B.10$b$}$$

To identify the form of $N$ and $N^\dagger$, we anticipate the result of (5.11) by bringing $e_0$ factors through all of the $H_i$ to the right in (B.10*a*) and to the left in (A.10*b*), giving

$$U = \tfrac{1}{2}(e_0 - e_3)\tfrac{1}{2}(e_0 - \mathrm{i}e_{123})\tfrac{1}{2}(\mathrm{i}e_4 + e_5)e_0e_0\mathrm{i}e_4 \tag{B.11$a$}$$

$$U = \tfrac{1}{2}\mathrm{i}e_4e_0e_0(\mathrm{i}e_4 - e_5)(e_0 + \mathrm{i}e_{123})\tfrac{1}{2}(e_0 + e_3). \tag{B.11$b$}$$

The expressions (A.11) provide the factorization of $U$ into nilpotents $U = NN^\dagger$, where $N$ and $N^\dagger$ are

$$\begin{aligned} N &= \tfrac{1}{8}(e_0 - e_3)(e_0 - \mathrm{i}e_{123})(\mathrm{i}e_4 + e_5) \\ N^\dagger &= \tfrac{1}{8}(\mathrm{i}e_4 - e_5)(e_0 + \mathrm{i}e_{123})(e_0 + e_3). \end{aligned} \tag{B.12}$$

Again, if we assume that the terms (B.6) are generated by the gauge transformation, contributing in the 'usual' way to the interaction, then the additional term arising because of the two-sided transformation is

$$Ng'h_-\rho_3 W_{4\mu} N^\dagger = -g' I W_{4\mu} N N^\dagger.$$

Once more, this term completes the lepton interaction (B.5).

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